Kuwait University
Department of Mathematics and Computer Science

Math 102 : Calculus II	Summer Semester 2009	In-Term Test 1

Saturday 18 July 2009

Duration: 90 minutes | Total marks: 25

Justify all your answers

1. Consider $f(x) = |x| \tan^{-1} x$ for $-\infty < x < \infty$.
   (a) Show that f is one-to-one on $(-\infty, \infty)$. [2 marks]
   (b) Determine the domain of f^{-1}. [1 mark]
   (c) Find $(f^{-1})'(\pi/4)$. [2 marks]
2. Prove that $\ln(1/x) = -\ln x$ for all $x > 0$. [2 marks]
3. Solve $\log_6(4-x) = 1 - \log_6(x+3)$. [2 marks]
4. Use logarithmic differentiation to find dy/dx when
$$y = \frac{x \exp(x^2)}{(e^{3x}+1)(\sin^2 x)\sqrt{x+1}}.$$ [2 marks]
5. Evaluate the following. [3 marks each]
   (a) $\displaystyle\int_{e^{-2}}^{e^{-1}} \frac{1}{t \ln t}\, dt$.
   (b) $\displaystyle\int \frac{1+\ln x}{2 - 3x\ln x}\, dx$.
   (c) $\displaystyle\int \frac{2x+1}{4x^2+1}\, dx$.
   (d) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.
6. Find the value of $\cos^{-1}(\sin(5\pi/4))$. [1 mark]
7. Prove that $\cosh 2x = \cosh^2 x + \sinh^2 x$, using the definition of hyperbolic functions. [1 mark]

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ANSWERS

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1. (a) $f(x) = x \arctan x$ and $f'(x) = \arctan x + x/(1+x^2)$ when $x > 0$, while $f(x) = -x \arctan x$ and $f'(x) = -\arctan x - x/(1+x^2)$ when $x < 0$. So, $f(x) > 0$ and $f'(x) > 0$ for $x > 0$, $f(0) = 0$, and, $f(x) < 0$ and $f'(x) > 0$ for $x < 0$. Hence, no horizontal line intersects the graph of $y = f(x)$ more than once.

   (b) As $x \to \pm\infty$, $\arctan x \to \pm\pi/2$, and hence $f(x) \to \pm\infty$, respectively. Since f is continuous on $(-\infty, \infty)$, this implies that the range of f is $(-\infty, \infty)$. Therefore the domain of f^{-1} is $(-\infty, \infty)$.

   (c) When $x = 1$, $f(x) = \pi/4$. So $f^{-1}(\pi/4) = 1$ and

$$\begin{aligned}(f^{-1})'(\pi/4) &= \frac{1}{f'(f^{-1}(\pi/4))} = \frac{1}{f'(1)} = \frac{1}{\arctan 1 + 1/(1+1^2)} \\ &= \frac{1}{\pi/4 + 1/2} = \frac{4}{\pi+2}.\end{aligned}$$

2. By definition,

$$\ln(1/x) = \int_1^{1/x} \frac{dt}{t}.$$

   Substituting $t = 1/u$, so $dt = (-1/u^2)\,du$, gives

$$\ln(1/x) = -\int_1^x \frac{du}{u} = -\ln x.$$

3. $\log_6(4-x) = 1 - \log_6(x+3) \implies \log_6\{(4-x)(x+3)\} = 1 \iff (4-x)(x+3) = 6 \iff x^2 - x - 6 = 0 \iff (x-3)(x+2) = 0 \iff x = 3$ or $x = -2$.
   Since, $4 - x > 0$ and $x + 3 > 0$ when $x = 3$ or $x = -2$, both are valid solutions.

4.

$$\ln|y| = \ln|x| + x^2 - \ln(e^{3x}+1) - 2\ln|\sin x| - \tfrac{1}{2}\ln(x+1)$$

$\implies$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2x - \frac{3e^{3x}}{e^{3x}+1} - 2\cot x - \frac{1}{2(x+1)}$$

$\implies$

$$\frac{dy}{dx} = \left\{2x + \frac{x+2}{2x(x+1)} - \frac{3e^{3x}}{e^{3x}+1} - 2\cot x\right\} \frac{x\exp(x^2)}{(e^{3x}+1)(\sin^2 x)\sqrt{x+1}}.$$

5. (a) Substitute $u = \ln t$. So $du = (1/t)\,dt$. Then

$$\int_{e^{-2}}^{e^{-1}} \frac{1}{t \ln t}\,dt = \int_{-2}^{-1} \frac{1}{u}\,du = \ln|u| \Big|_{-2}^{-1} = \ln|-1| - \ln|-2| = -\ln 2.$$

(b) Substitute $u = 2 - 3x\ln x$. So $du = -3(\ln x + 1)\,dx$. Then

$$\begin{aligned} \int \frac{1+\ln x}{2 - 3x\ln x}\,dx &= -\frac{1}{3}\int \frac{1}{u}\,du = -\tfrac{1}{3}\ln|u| + C \\ &= -\tfrac{1}{3}\ln|2 - 3x\ln x| + C. \end{aligned}$$

(c)
$$\begin{aligned} \int \frac{2x+1}{4x^2+1}\,dx &= \int \frac{2x}{4x^2+1}\,dx + \frac{1}{4}\int \frac{1}{x^2 + (1/2)^2}\,dx \\ &= \tfrac{1}{4}\ln(4x^2+1) + \tfrac{1}{2}\arctan(2x) + C. \end{aligned}$$

(d) Substitute $x = -\ln u$. So $dx = (-1/u)\,du$. Then

$$\begin{aligned} \int \frac{dx}{\sqrt{e^{2x}-1}} &= -\int \frac{1}{u\sqrt{u^{-2}-1}}\,du = -\int \frac{1}{\sqrt{1-u^2}}\,du = \arccos u + C \\ &= \arccos(e^{-x}) + C. \end{aligned}$$

An alternative answer is $-\arcsin u + C = -\arcsin(e^{-x}) + C$. Alternative substitutio are $x = \ln u$ leading to $\sec^{-1} u + C$ or $-\csc^{-1} u + C$ with $u = e^x$, or, $x = \frac{1}{2}\ln(u^2 +$ leading to $\arctan u + C$ or $-\cot^{-1} u + C$ with $u = \sqrt{e^{2x}-1}$.

6.
$$\arccos(\sin(5\pi/4)) = \arccos(-1/\sqrt{2}) = 3\pi/4.$$

Alternatively,

$$\begin{aligned} \arccos(\sin(5\pi/4)) &= \pi/2 - \arcsin(\sin(5\pi/4)) = \pi/2 - \arcsin(\sin(-\pi/4)) \\ &= \pi/2 - (-\pi/4) = 3\pi/4. \end{aligned}$$

7.
$$\begin{aligned} \cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} \\ &= \cosh 2x. \end{aligned}$$